Where is the primal Total Factor Productivity Index?

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Abstract: A primal index of productivity change is introduced which decomposes exactly in three components: technical change, technical efficiency change and average scale economies (radial scale change). The proposed index is invariant to movement along indifference surfaces and it collapses to the Malmquist index if the technology is locally constant returns to scale. It is the best linear approximation of the Orea (2002) translog productivity index (primal growth accounting). The existing proposals for the primal productivity index are discussed in detail. An empirical illustration on real data is provided to show the magnitude of the residuals associated to the lack of indifference surface invariance for these indexes.

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JEL Classification: D24
1. Introduction

In the last 20 years productivity scholars oriented their research to find general\(^1\) primal definitions of total factor productivity growth. In spite of this effort, there is still disagreement about the definition of this measure. Caves et al (1982) introduced the Malmquist productivity index (CCD Malmquist from now on) as a way to define the primal productivity measure. This measure of productivity change accommodates for multi-output multi-input production processes and does not require the definition of a specific functional form for the production technology. The advantage of the Malmquist approach is that it is intrinsically nonparametric, requiring only a very minimal set of axioms to be satisfied by the production technology. The CCD Malmquist productivity index has been shown to be decomposable into two components: technical change (shift in the production frontier) and technical efficiency change (catch-up with the production frontier). The Malmquist approach was popularized as an applied tool by Fare et al (1994) which used the Data Envelopment Analysis (DEA) methodology to estimate directly the production technology and, therefore, the primal Malmquist productivity index. Since then a growing literature used to estimate productivity in a primal approach using the Malmquist index. In fact, a primal setting permits to investigate productivity performance also in cases where price information is unavailable and index numbers cannot be used. Unfortunately, as Grifell-Tatje and Lovell (1995) pointed out clearly, the Malmquist productivity measure does not account for average scale economies. Thus (unless the constant returns to scale assumption is met) the CCD Malmquist measure neglects a potentially important contribution to productivity growth. Many authors tried to develop a more general measure able to account also for scale economies. These attempts produced a variety of primal productivity measures which usually returns different numbers for the same data. This situation is quite unsatisfactory, especially from an empirical/applied point of view.

Three main proposals have come to prominence in the literature. First, Fare et al (1994) proposed to measure productivity computing the CCD Malmquist index against a virtual CRS cone technology\(^2\) (FGNZ index from now on). This procedure has been advocated also by Ray and Desli (1997), although with a different decomposition. Later on Balk (2001, 2004) argued in favour of this approach, discussing a number of alternative decompositions of the index\(^3\). Second, Grifell-Tatje and Lovell (1999) (GTL from now on) proposed to generalize the CCD Malmquist index multiplying it by a scale economies

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\(^1\) The attempts made in the literature rely on very general definitions of technology which are based on axiomatic production theory (and its functional representation via distance functions).

\(^2\) The virtual CRS cone technology is an enlargement of the actual technology. See next section for formal definitions.

\(^3\) Grosskopf (2003) and Lovell (2003) discussed alternative decompositions of the index.
component. Third, Bjurek (1996) proposed the ratio of a Malmquist output quantity change to a Malmquist input quantity change. All these indexes identify the same structural components (technical change, technical efficiency change, average scale economies) and differ for a residual component that represents lack of invariance of the productivity measure to movements along indifference surfaces (isoquants). It is argued that this residual component has no economic interpretation and it does not correspond to a meaningful component of productivity change. To illustrate intuitively the meaning of these residuals, consider a situation where a firm is technical efficient, there is no technical change, the output vector is constant and the input vector move along the pre-assigned input isoquant defined by the fixed output vector. In this special case there is no productivity change⁴, still these productivity measures return a number different from 1.

In this study a primal total factor productivity definition is provided that decomposes exactly in the three aforementioned components (technical change, technical efficiency change and average scale economies) without leaving unexplained residuals. The productivity index is nonparametric in nature and defined using variation of the distance function along pre-assigned rays. For this reason it is deemed radial productivity index (RPI). It is shown that when the technology is truly translog, the new index is the best linear approximation of the translog productivity index proposed by Orea (2002). In this respect the RPI index represents a nonparametric generalization of the growth accounting procedure. The RPI has the same computational complexity of existing indexes and it collapse to the CCD Malmquist index if the technology is truly CRS. Therefore it incorporates average scale economies, yet it is invariant to movement along indifference surfaces (isoquants). This is a nice property that is exploited in order to better understand the nature of the residuals embedded in the three alternative productivity indexes (FGNZ, GTL and Bjurek). These residuals are found to have no economic interpretation and to be responsible for the lack of invariance to movements along indifference surfaces. All the theoretical findings are illustrated using an agricultural dataset provided by Coelli and Rao (2005). As shown in the empirical exercise the magnitude of the residuals can be very large and overwhelm the productivity growth signal.

The reminder of the paper is organized into sections. In section 2 the axiomatic technology and its properties are introduced. Section 3 discusses the primal translog productivity measure and the Törnqvist growth accounting methodology. Section 4 is dedicated to the building of the radial productivity index.

⁴ The example considers a change of the input mix along the isoquant. This is a movement along an indifference surface therefore the aggregate measure (productivity in a production context) should be invariant. If one wishes to evaluate different points along a given isoquant, price considerations and behavioural assumption have to be bring in the analysis.
Section 5 reviews the main primal productivity indexes proposed in the literature. In section 6 a brief empirical illustration based on a real dataset is provided. Finally, section 7 concludes.

### 2. Background: axiomatic production theory

The modern approach to define primal total factor productivity indexes is based on axiomatic production theory. Production theory accommodates for very general technology definitions, requiring only a limited number of regularity conditions (axioms). In this section the basic definitions and properties are sketched out. Let consider a production process that produces $y \in R^M$ outputs by means of $x \in R^N$ inputs. Inputs and outputs are assumed to be measurable and continuous variables, although discrete variables can be accommodated if needed. The production set or technology set is the set of all the feasible production plans: $T = \{(x,y) \in R^N x R^M : x \text{ can produce } y\}$. Minimal standard regularity conditions on $T$ are (see Fare and Primont, 1995): 1) inactivity is possible: $(x,0,y) \in T$; 2) strong disposability of inputs and outputs; 3) scarcity, i.e. for any given input vector only a limited quantity of outputs is producible; 4) no free lunch: $(0,y) \notin T$. It is not in general assumed that the production set is convex, although in some parts of the paper this will be explicitly mentioned. These are quite general axioms and can be further relaxed if one wishes. For the purposes of this study, these axioms are regarded as sufficiently general in order to define a primal total factor productivity index. In fact, the production set defined by these set of axioms is general enough to encompass all the proposals. The purpose of the next sections is to establish a primal measure of total factor productivity based on this very general framework. It should be noted that this framework can accommodate a large variety of situations, from multi-output technology to non differentiable production functions, from non-competitive markets to non optimizing behaviour. Before going ahead with definitions of productivity let introduce some further notation. Under the circumstances defined above, the production set may be also represented by the input and output sets. The output set is the collection of all the output vectors producible by a given input quantity vector $P(x) = \{y \in R^M : (x,y) \in T\}$ and can be represented in a functional form by the output distance function:

$$D_o(x,y) = \min \left\{ \theta > 0 : \frac{y}{\theta} \in P(x) \right\}$$

The input set is the collection of all the input vectors able to produce a given output quantity vector $L(y) = \{x \in R^N : (x,y) \in T\}$ and can be represented in a functional form by the input distance function:
\[ D_i(x, y) = \max \{ \lambda > 0 : (\lambda x, \lambda y) \in L(y) \} \]

Properties of the distance functions can be derived by the axioms of technology. Of particular importance are the homogeneity properties of these functions: the output distance function is linearly homogeneous in outputs and the input distance function is linearly homogeneous in inputs. A technology set is said to be homogeneous if and only if:

\[ \forall (x, y) \in T \rightarrow (\lambda x, \lambda^\alpha y) \in T, \lambda > 0 \]

The last definition implies that (for homogeneous technologies) the output distance function can be recovered by the input distance function (and vice versa): \( D_o(x, y) = \frac{1}{[D_i(x, y)]^\alpha} \). For \( \alpha = 1 \) the constant returns to scale (CRS) technology is considered. Associated to the actual production set \( T \) it is possible to define the virtual CRS cone technology \( T_v = \{(\lambda x, \lambda^\alpha y) : (x, y) \in T, \lambda > 0\} \) as an enlargement of the actual production set. This virtual technology can be characterized in a functional form by the virtual CRS output distance function \( D_o(x, y) \) and the virtual CRS input distance function \( D_i(x, y) \) (where the subscript “c” means that the distance function is computed against the virtual CRS technology). The virtual CRS technology allows defining output and input scale efficiencies as \( SE_o(x, y) = \frac{D_\infty(x, y)}{D_o(x, y)} \) and \( SE_i(x, y) = \frac{D_o(x, y)}{D_i(x, y)} \). The technology satisfies input and output homotheticity if (respectively):

\[ D_i(x, y) = \frac{1}{H(y)} D_i(x, 1) \]

\[ D_o(x, y) = \frac{1}{G(x)} D_o(1, y) \]

where \( H(y) \) and \( G(x) \) are consistent with the technology axioms. One of the consequences of homotheticity is that (when the distance function is differentiable) output elasticities are independent from

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5 The virtual CRS cone technology exists only if one assumes some other restrictions on the production set. For example, it does not exist in the case of globally increasing returns to scale. Another example of infeasibility is given by technologies satisfying the Inada conditions; in this special case the virtual CRS cone technology corresponds to the all positive orthant (therefore it is a degenerate case). The reason why this special virtual technology is introduced in the discussion is that some of the proposals for the primal productivity index rely on the specification of this virtual technology.
inputs and input elasticities are independent from output (it is a form of separability). For example, assuming output homotheticity one obtains:

\[ \varepsilon_m = \frac{\partial \ln D_o(x, y)}{\partial \ln y_m} = \frac{\partial \ln D_o(1, y)}{\partial \ln y_m} = \varepsilon_m(y), \quad \forall m = 1, \ldots, M \]

\[ \varepsilon_n = \frac{\partial \ln D_o(x, y)}{\partial \ln x_n} = \frac{\partial \ln G(x)}{\partial \ln x_n} = \varepsilon_n(x), \quad \forall n = 1, \ldots, N \]

If one assumes, additionally, that \( D_i(x, y) = 1 \) if and only if \( D_o(x, y) = 1 \), joint input and output homotheticity imply inverse homotheticity (see Fare & Primont, 1995b):

\[ D_o(x, y) = \frac{D_o(1, y)}{F(D_i(x, 1))} \]

where \( F(\cdot) \) is a transform (it is increasing, invertible and continuous, with \( F(0) = 0 \) and \( F(v) \to \infty \) for \( v \to \infty \)). Assuming revenue maximizing behaviour the output set can be characterized by the revenue function\(^8\): \( R(x, p) = \max_y \{py : y \in P(x)\} \). Assuming cost minimizing behaviour the input set can be characterized by the cost function\(^9\): \( C(w, y) = \min_x \{wx : x \in L(y)\} \).

The definition of a productivity measure corresponds to the output oriented comparison of two firms: the base period firm \( (x^t, y^t) \) facing technology \( T^t \) and the comparison period firm \( (x^{t+1}, y^{t+1}) \) facing technology \( T^{t+1} \). Let assume that the analyst has complete knowledge of the technology in the two time periods and that the technology is represented by the output distance function. The representation here provided is perfectly general, since the two firms can be the same firm in two different periods of

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\(^6\) Here it is assumed that the distance function is differentiable. This implies a more restrictive production set than the one defined by the axioms. As a matter of fact, differentiable distance functions are a special class of the technology sets considered in this study. In many parts of this paper differentiability is assumed as an additional hypothesis, although the final definition of productivity does not rely on this assumption, but only on the technology axioms.

\(^7\) This assumption is not, in general, needed for defining the primal productivity index. Nevertheless it is convenient to use it in some parts to simplify notation. This assumption is not met, for example, by production models that use Data Envelopment Analysis (DEA) techniques.

\(^8\) It should be noted that for the revenue function to be well defined one has to assume a convex output set, neoclassical perfectly competitive markets and optimizing (revenue maximizing) behaviour of the agents. In general the primal definition of productivity should be able to avoid all these assumptions.

\(^9\) For a well behaved cost function, it has to be assumed a convex input set, neoclassical perfectly competitive markets and cost minimizing behaviour.
time (time series), two firms at the same period of time (cross section) or two firms in two different
periods of time. In this paper the time series language will be used although it is only a matter of
convenience. All the analysis will be presented using the output orientation (output distance function). Of
course, extension to the input orientation mirrors the discussion of the output orientation, thus it adds only
length, not content, and will be here not discussed further.

3. Growth accounting: the output translog technology and the
Tornqvist Index

In this section a very special technology is discussed: the output translog technology. This is a
quite restrictive technology, since assumes a specific functional form for the output distance function. For
this special case a very uncontroversial measure of productivity based on differential calculus exists. It
should be noted that in the special case of an output translog technology the primal productivity measure
can be interpreted as a growth accounting exercise. The change in the distance function between the two
periods of time corresponds to what one usually refers to as technical efficiency change (TEC):

\[
\ln TEC = \ln D_o(x^{t+1}, y^{t+1}, t+1) - \ln D_o(x', y', t)
\]

Following Lovell (2003) and Orea (2002), for the translog functional form (quadratic in logs) one
can use the quadratic identity lemma (Diewert, 1976) or the translog identity (Caves et al, 1982) to
determine the exact log change in the distance function and correctly impute it to each variable:

\[
\begin{align*}
\left[ \ln D_o(x^{t+1}, y^{t+1}, t+1) - \ln D_o(x', y', t) \right] &= \\
&= \frac{1}{2} \sum_m \left[ \varepsilon_m^{t+1} + \varepsilon_m' \right] \ln y_m^{t+1} - \ln y_m' + \\
&+ \frac{1}{2} \sum_n \left[ \varepsilon_n^{t+1} + \varepsilon_n' \right] \ln x_n^{t+1} - \ln x_n' + \\
&+ \frac{1}{2} \left[ \frac{\partial \ln D_o(x^{t+1}, y^{t+1}, t+1)}{\partial t} + \frac{\partial \ln D_o(x', y', t)}{\partial t} \right] \Delta t
\end{align*}
\]

(1)

where

\[
\varepsilon_n' = \varepsilon_n(x', y', t) = \frac{\partial \ln D_o(x', y', t)}{\partial \ln x_n}, \; \forall n = 1, \ldots, N
\]
The notation emphasizes that the input and output elasticities depend on the input-output vector and the technology (time varying). Equation (1) provides a way to identify the contribution of input change, output change and technical change (time change) to the technical efficiency change. Since the output distance function is homogeneous of degree +1 in outputs (in each period of time), first order derivatives with respect to output sum up to one:

\[ \sum_m \varepsilon_n^m = 1 \]

Input derivatives sum up to the scale elasticity:

\[ \sum_n \varepsilon_n^m = \varepsilon^m \].

The output contribution in equation (1) provides also an output aggregator function that is linearly homogeneous:

\[ \Delta \ln Y = \frac{1}{2} \sum_m \left[ \varepsilon_n^m + \varepsilon^m \right] \left( \ln y_n^m + \ln y^m \right) \]  

(2)

On the contrary input change in equation (1) does not provide a proper aggregator function for inputs (it is not linearly homogenous), but it can be decomposed in an aggregate input change and an average scale economies change:

\[ \Delta \ln X = -\frac{1}{2} \sum_n \left[ \varepsilon_n^x + \varepsilon^x \right] \left( \ln x_n^x + \ln x^x \right) \]  

(3)

\[ \ln S \Delta = -\frac{1}{2} \sum_n \left[ (\varepsilon_n^x - 1) \frac{\varepsilon_n^x}{\varepsilon^x} + (\varepsilon^x - 1) \frac{\varepsilon_n^x}{\varepsilon^x} \right] \left( \ln x_n^x - \ln x^x \right) \]  

(4)

The first equation provides a linearly homogenous aggregator function for input change and the second one is a measure of average scale economies. Rearranging terms in equation (1) and using the

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10 Linear homogeneity is normally considered an important property that the inputs and outputs aggregator functions should satisfy.
definition of average scale economies one obtains a primal measure of productivity and its decomposition:

\[
\Delta \ln Y - \Delta \ln X = \frac{1}{2} \sum_n \left[ \left( e^{rt+1} - 1 \right) \frac{e^{rt+1}}{e^{rt}} + \left( e^{rt} - 1 \right) \frac{e^{rt}}{e^{rt}} \right] \left( \ln x^{rt+1}_o - \ln x^t_o \right) + \\
- \frac{1}{2} \left[ \frac{\partial \ln D_o(x^{rt+1}, y^{rt+1}, t+1)}{\partial t} + \frac{\partial \ln D_o(x^t, y^t, t)}{\partial t} \right]
\]

(5)

Equation (5) corresponds to the primal growth accounting procedure and it identifies three sources of productivity growth on the right hand side: first, the contribution of average scale economies \((\ln S\Delta)\); second, the technical change component \((\ln TC)\); third, the change in technical efficiency \((\ln TEC)\). These components of productivity are non-controversial, in the sense that they are based on calculus. In other words, the use of differentiation permits to identify the contribution of each variable to the change in the distance function\(^{11}\). The other way of defining productivity change (in the spirit of the Malmquist tradition) is using ratios of distance functions computed across mixed period observations (it is a linear averaging procedure). This is equivalent to keep some of the variables constant while moving the others in the attempt to separate the contribution of each single variable. This approach is, of course, in sharp contrast to calculus, where the contribution of each single variable is accounted for considering also the movement in the other variables (roughly speaking). The distance function ratio procedure became popular after the Caves et al (1982) breakthrough theoretical paper on the Malmquist productivity index. The approach found a large echo in applied research thanks to the seminal paper of Fare et al (1994) that was able to estimate the distance function directly in a primal framework (nonparametrically). The advantage of using ratios of distance functions is that the productivity measure is still defined if the output distance function is not differentiable and of the translog form (nonparametric approach). Of course, it is desirable that productivity defined as a ratio of distance functions and productivity defined on calculus equalize each other when the technology is truly translog. Therefore expression (5) can help in demarcating between alternative proposals for the primal productivity index.

\(^{11}\)This procedure is specific to the translog or quadratic specification. In general, assuming only differentiability, one needs to use integration to obtain the exact change in the output distance function and impute it to each variable. Therefore a simple imputation like (5) for discrete time change is possible only for translog and quadratic distance functions. Although these functional specifications are more restrictive than the axiomatic production framework, they are the best second order approximation of any twice differentiable functional specification.
A very common practice in the growth accounting tradition is to use cost and revenue shares to aggregate inputs and outputs. This procedure finds a rationale justification in what follows. The dual to the productivity measure defined by equation (5) can be obtained assuming revenue maximizing behaviour and neoclassical perfect competition (price taker agents). Under these assumptions the technical efficiency change component vanishes; the scale and technical change components can be identified by the derivatives of the revenue function, therefore defining implicitly an index of output change. The gradient of the revenue function with respect to output prices gives output shares:

\[
s_m = \frac{\partial \ln R(x', p', t)}{\partial \ln p_m} = \frac{\partial \ln D_o(x', y', t)}{\partial \ln y_m}
\]

The sum of the derivatives with respect to inputs gives the output scale elasticity:

\[
\varepsilon_o = -\sum_n \frac{\partial \ln R(x', p', t)}{\partial \ln x_n} = \sum_n \frac{\partial \ln D_o(x', y', t)}{\partial \ln x_n}
\]

Through the output scale elasticity, one is able to compute the input aggregator shares:

\[
\varepsilon_o' = -\frac{\partial \ln R(x', p', t)}{\partial \ln x_n} = \frac{\partial \ln D_o(x', y', t)}{\partial \ln x_n}
\]

If, moreover, cost minimizing behaviour is assumed, the shares correspond to observed cost shares\(^{12}\):

\[
\frac{\varepsilon_o'}{\varepsilon_o} = -\frac{\partial \ln R(x', p', t)}{\partial \ln x_n} = \frac{\partial \ln D_o(x', y', t)}{\partial \ln x_n} = s_m = \frac{\partial \ln C(w', y', t)}{\partial \ln w_n}
\]

Substitution of the revenue and cost shares into the original index number gives the following formula:

\[
\Delta \ln Y - \Delta \ln X = \frac{1}{2} \sum_m [s_m^{t+1} + s_m^t] (\ln y_m^{t+1} - \ln y_m^t) - \frac{1}{2} \sum_n [s_n^{t+1} + s_n^t] (\ln x_n^{t+1} - \ln x_n^t)
\]

\(^{12}\) Another common way of getting this result is by assuming a CRS technology. In this case \(\varepsilon_o = 1\) and the cost shares equalize derivatives of the revenue function. This alternative way is specially used in macroeconomics, where the CRS assumption is more justifiable.
where $s_m^t = \frac{p_m^t y_m^t}{p^t y^t}$ is the share of the m-th output at time t and $s_n^t = \frac{w_n^t x_n^t}{w^t x^t}$ is the share of the n-th input at time t. Equation (6) is the ratio (log difference) of a Tornqvist output quantity change to a Tornqvist input quantity change, thus a Tornqvist index of productivity change. The Tornqvist productivity index deviates from the primal productivity index (5) if the duality assumptions do not hold (optimizing behavior, full technical efficiency and perfect competition). Therefore the Tornqvist index can be regarded as the dual of the productivity index (5). Under these conditions the Tornqvist index is equal to the primal index of productivity change defined by (5). It is worth noting that these conditions are more general than the sufficient conditions provided by Caves el (1982). More specifically, in this framework the translog distance function in the two periods of time can have different second order parameters. Equation (6) is a standard growth accounting exercise, where productivity growth is defined as the difference between aggregate output growth and aggregate input growth. Computation of equation (6) requires knowledge of prices and assumption of neoclassical perfect competition and optimizing behaviour. When these conditions are not met equation (5) can be used. Equation (5) is a primal growth accounting exercise based on a translog functional specification. The purpose of the next sections is to build an index of productivity which is based only on the basic axioms of technology and to evaluate the performance of alternative proposals for the primal productivity index. This index of productivity should be interpreted as a generalization of the growth accounting exercise; therefore care will be taken to show the relationship with the primal growth accounting (5).

4. A radial total factor productivity index

In this section a nonparametric total factor productivity index based on ratios of distance functions is presented. Sufficient conditions (on the production set) for the index to be equal to the translog productivity index (5) are provided. These conditions turn out to be very restrictive. It is shown that this productivity index is the best linear approximation of the translog index of equation (5). The index is decomposable as a CCD Malmquist index by a radial scale change component that proxies average scale economies. For this reason the index is deemed radial. As mentioned in the introduction the building blocks for any Hicks-Moorsteen type productivity index are the Malmquist input and output indexes of quantity change. These indexes date back at least to Caves et al. (1982) and Diewert (1992). A Malmquist

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\[\text{This point is worth noting. The introduction of a general nonparametric definition of productivity change based on the minimal axioms of production theory should improve on the growth accounting. Therefore the primal index one is searching for should collapse, or at least approximate the growth accounting exercise when the technology is truly translog.}\]
output quantity index is defined evaluating the change in the output distance function conditionally to a reference technology and a given input vector $\mathbf{x}$:

$$
Y(y', y'^{it} | \mathbf{x}, t) = \frac{D_y(x, y'^{it}, t)}{D_y(x, y', t)}
$$

where the notation emphasizes that the Malmquist output quantity index is conditional to the choice of the input vector (conditioning input vector) and of the reference technology. The Malmquist output quantity index becomes independent from the choice of the reference technology if technical change is Hicks neutral. In this special case $Y(y', y'^{it} | \mathbf{x}, t) = Y(y', y'^{it} | \mathbf{y})$. The index is independent from the choice of the conditioning input vector when the technology is output homothetic, in which case $Y(y', y'^{it} | \mathbf{x}, t) = Y(y', y'^{it} | t)$. When the technology is output homothetic and technical change is Hicks neutral, the Malmquist output quantity index is independent from the choice of the input conditioning vector and the reference technology: $Y(y', y'^{it} | \mathbf{x}, t) = Y(y', y'^{it})$. A Malmquist input quantity index is defined evaluating the change in the input distance function conditionally to a reference technology and a given output vector:

$$
X(x', x'^{it} | \mathbf{y}, t) = \frac{D_x(x'^{it}, \mathbf{y}, t)}{D_x(x', \mathbf{y}, t)}
$$

The Malmquist input quantity index is independent from the choice of the reference technology if technical change is Hicks neutral ($X(x', x'^{it} | \mathbf{y}, t) = X(x', x'^{it} | \mathbf{y})$). It is independent from the choice of the output conditioning vector if the technology is input homothetic and in this special case: $X(x', x'^{it} | \mathbf{y}, t) = X(x', x'^{it} | t)$. When technology is input homothetic and technical change is Hicks neutral, the Malmquist input quantity index is invariant to the choice of the reference technology and the conditioning output vector: $X(x', x'^{it} | \mathbf{y}, t) = X(x', x'^{it})$. Productivity is defined as a ratio of an output Malmquist quantity indexes to an input Malmquist quantity indexes. This type of productivity indexes have been named as “Hicks-Moorsteen” after Diewert (1992) and Fare et al (1996) advocated the idea to Moorsteen and Hicks. In the very special case when the technology is inversely homothetic and technical change is Hicks neutral a unique uncontroversial measure of productivity arises:

$$
TFP(x', x'^{it}, y', y'^{it}) = \frac{Y(y', y'^{it})}{X(x', x'^{it})}
$$
When these very restrictive assumptions do not hold, the question arises of what reference technology and which conditioning input-output vector to use. In this section it will be shown that the following index defined as the ratio of Malmquist quantity indexes has nice properties in approximating the translog index (5). The index is defined as

\[ RPI = \left( \frac{Y'(y',x'^{r+1})}{X'(x',x'^{r+1})} \right)^{\frac{1}{2}} \left( \frac{Y'(y',y'^{r+1}|x',t+1)}{X'(x'^{r+1}, x'^{r+1}|y',t+1)} \right)^{\frac{1}{2}} \]  

(7)

This productivity index is the geometric mean of two productivity indexes both defined as ratios of Malmquist quantity indexes: the base period productivity index and the comparison period productivity index. The Malmquist output quantity index is defined as the geometric mean of two sub-indexes:

\[ Y = \left[ Y'(y',y'^{r+1}|x',t)Y'(y',y'^{r+1}|x'^{r+1},t+1) \right]^{\frac{1}{2}} = \left[ \frac{D_o(x'^{r+1},y'^{r+1},t)D_o(x',y'^{r+1},t+1)}{D_o(x',y'^{r+1},t)D_o(x',y',t+1)} \right]^{\frac{1}{2}} \]  

(8)

The Malmquist input quantity index is defined as the geometric mean of two sub-indexes:

\[ X = \left[ X'(x',x'^{r+1}|y',t)X'(x',x'^{r+1}|y'^{r+1},t+1) \right]^{\frac{1}{2}} = \left[ \frac{D_i(x'^{r+1},y'^{r+1},t)D_i(x',y'^{r+1},t+1)}{D_i(x',y'^{r+1},t)D_i(x',y',t+1)} \right]^{\frac{1}{2}} \]  

(9)

where the conditioning output vectors are defined as:

\[ y'^{r+1} = \frac{y'^{r+1}}{D_o(x'^{r+1},y'^{r+1},t+1)} \text{ and } y'^{r+1} = \frac{y'^{r+1}}{D_i(x'^{r+1},y'^{r+1},t)} \]

This index will be called the radial productivity index (RPI) because it incorporates a radial scale change measure (invariant to input and output mix changes) by a CCD Malmquist productivity index (which is radial in nature). The product of technical change (TC) by technical efficiency change (TEC) returns the CCD Malmquist productivity index:

\[ M(x',y',x'^{r+1},y'^{r+1}) = \frac{D_o(x'^{r+1},y'^{r+1},t+1)}{D_o(x',y',t)} \left[ \frac{D_o(x',y',t)}{D_o(x',y',t+1)} \frac{D_o(x'^{r+1},y'^{r+1},t)\right]^{\frac{1}{2}} = TEC \cdot TC \]  

(10)

and the radial scale change component (RSC) is given by the following expression:
\[ RSC = \left( RSC^t RSC^{t+1} \right)^{\frac{1}{2}} = \left( \frac{1}{\lambda} D'_o(x_i, y_i) \frac{1}{\lambda} D'_o^{t+1}(x_i, y_{t+1}) \right)^{\frac{1}{2}} \] (11)

where \( X(x', x^{t+1} | y', t) = \lambda' \) and \( X(x', x^{t+1} | y^{t+1}, t+1) = \lambda^{t+1} \) for easy of notation\(^{14}\). The index identifies only average radial scale change components without leaving unexplained input and output mix residuals.

In this section it is proven that the CCD measure of technical change is the best linear approximation of the translog technical change component of equation (5) and the radial scale change is the best linear approximation of the average scale economies of equation (4).

**The CCD Malmquist index**

Caves et al (1982) proposed the Malmquist productivity index to identify the technical efficiency and technical change components. The technical efficiency change (TEC) component is directly observable as the change in the value of the output distance function and this is consistent with the Malmquist definition. On the contrary, unless technical change is Hicks neutral, the Malmquist definition of technical change (or any other definition based on the average distance function variation along a pre-assigned input-output ray) misspecifies the technical change component of equation (5) in the case of a translog technology. Nevertheless it represents its best linear approximation as it is proven in the next propositions.

Note that under Hicks neutrality the output distance function can be written as: \( \ln D_o(x, y, t) = \ln A(t) + \ln D_o(x, y) \). By simple differentiation one obtains that input and output elasticities are time invariant:

\[
\varepsilon_m(x, y, t+1) = \varepsilon_m(x, y, t) = \varepsilon_m(x, y), \quad \forall m = 1, \ldots, M
\]

\[
\varepsilon_n(x, y, t+1) = \varepsilon_n(x, y, t) = \varepsilon_n(x, y), \quad \forall n = 1, \ldots, N
\]

It is also possible to check that time elasticity is invariant with respect to inputs and outputs in the Hicks neutral case. In fact:

---

\(^{14}\) These radial scale changes can be written in terms of scale efficiency changes (if the associated virtual CRS cone technology exists) as: \( RSC^t = \frac{SE_o(x^{t+1}, y_{t+1}, t)}{SE_o(x_i, y_i, t)} \), \( RSC^{t+1} = \frac{SE_o(x^{t+1}, y_{t+1}, t+1)}{SE_o(x_i, y_i, t+1)} \).
Proposition 1: if technical change is Hicks neutral then the CCD Malmquist index of technical change is equal to the translog technical change component of equation (5).

Proof: the CCD Malmquist index of technical change is:

\[
\ln TC = \frac{1}{2} \left[ \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t) - \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t + 1) \right] + \frac{1}{2} \left[ \ln D_o(x', y', t) - \ln D_o(x', y', t + 1) \right]
\]

Applying the quadratic lemma to both these components one obtains:

\[
\ln D_o(x^{i^{*i}}, y^{i^{*i}}, t) - \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t + 1) = \frac{1}{2} \left[ \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t)}{\partial t} + \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t + 1)}{\partial t} \right]
\]

\[
\ln D_o(x', y', t) - \ln D_o(x', y', t + 1) = \frac{1}{2} \left[ \frac{\partial \ln D_o(x', y', t)}{\partial t} + \frac{\partial \ln D_o(x', y', t + 1)}{\partial t} \right]
\]

Therefore the Malmquist technical change expressed in terms of derivatives of the translog output distance function is

\[
\ln TC = \frac{1}{4} \left[ \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t)}{\partial t} + \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t + 1)}{\partial t} \right] + \frac{1}{4} \left[ \frac{\partial \ln D_o(x', y', t)}{\partial t} + \frac{\partial \ln D_o(x', y', t + 1)}{\partial t} \right]
\]

This expression corresponds to the translog technical change component identified by equations (5) if and only if:

\[
\left[ \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t)}{\partial t} - \frac{\partial \ln D_o(x', y', t)}{\partial t} \right] + \left[ \frac{\partial \ln D_o(x^{i^{*i}}, y^{i^{*i}}, t + 1)}{\partial t} - \frac{\partial \ln D_o(x', y', t + 1)}{\partial t} \right] = 0
\]

It is easy to check that this equality hold when technical change is Hicks neutral.
Other sufficient conditions for the CCD Malmquist to correctly identify the technical change component in equation (5) have been provided by Caves et al (1982) and require that the second order parameters of the translog output distance function are time invariant. If technical change is not Hicks neutral (or the translog have time dependant second order coefficients), the Malmquist index of productivity change deviates from the translog index embedded in (5). It should also be emphasized that the bias is transmitted to the CCD Malmquist productivity index.

Proposition 2: The CCD Malmquist index of technical change is the best linear approximation of the translog technical change component of equation (5).

Proof: consider the factor of proportionality by which the output vector has to be increased so that the inflated output is still on the surface of the production frontier at time \( t + \varepsilon \): \( \mu(x, y, t + \varepsilon) \). The first component of technical change in equation (5) can be expressed using the difference quotients:

\[
\frac{\partial \ln D_o(x', y', t)}{\partial t} = \lim_{\varepsilon \to 0} \ln D_o(x', y', t + \varepsilon) - \ln D_o(x', y', t)
\]

Therefore a linear approximation is provided by:

\[
\frac{\partial \ln D_o(x', y', t)}{\partial t} \approx \ln D_o(x', y', t + \varepsilon) - \ln D_o(x', y', t)
\]

On the right hand side is the base period Malmquist index of technical change. If time periods \( t \) and \( t+1 \) are close enough, this is a good approximation of the time derivatives. With a similar approach one can derive the comparison period approximation.

Average scale economies

The second part of the radial productivity index is the radial scale change component. Sufficient conditions for this index to be equal to the translog average scale change index (4) are provided. Since these conditions are quite restrictive, it will be shown that the radial scale change component is the best linear approximation of the translog average scale economies component (4).

Proposition 3: the best linear approximation of the translog scale elasticity is, for a given \( \lambda > 0 \) :

\[
\lambda \ln D_o(x', y', t + \varepsilon) - \ln D_o(x', y', t)
\]
\[ \varepsilon \equiv \frac{\ln D_o(x, y) - \ln D_o(\lambda x, y)}{\ln \lambda} \]

**Proof:** The scale elasticity can be computed as follow. Suppose to rescale the input vector by \( \lambda > 0 \) and define the radial change in the output quantities \( \mu > 0 \) associated to \( \lambda > 0 \) such that: 
\[ \ln D_o(\lambda x, \mu y) = \ln D_o(x, y). \]
For the homogeneity conditions of the output distance function one obtains 
\[ \ln \mu(\lambda, x, y) = \ln D_o(x, y) - \ln D_o(\lambda x, y). \]
For any given \( \lambda > 0 \) one obtains 
\[ \ln \mu = \ln D_o(x, y) - \ln D_o(\lambda x, y) \]
and using the difference quotient one can define the elasticity of scale as:
\[ \varepsilon = \frac{\partial \ln \mu(\lambda, x, y)}{\partial \ln \lambda} = \lim_{\lambda \to 1} \frac{\ln \mu(\lambda, x, y) - \ln \mu(1, x, y)}{\ln \lambda} \]
This implies the stated approximation.

**Proposition 4:** the Malmquist input quantity change (9) is the best linear approximation of the translog aggregate input change (3).

**Proof:** For the base period (assuming for easy of notation \( X(x', x'^{i+1} | y', t) = \lambda' \))
\[ X(x', x'^{i+1} | y', t) = \lambda' = \frac{D_o(x'^{i+1}, y', t)}{D_o(x', y', t)} \]
implies 
\[ D_o(\lambda' x', y', t) = D_o(x'^{i+1}, y', t) = 1. \]
It follows: 
\[ D_o(\lambda' x', y', t) = D_o(x'^{i+1}, y', t) = 1 \] and 
\[ D_o(\lambda' x', y', t) = D_o(x'^{i+1}, y', t). \]
Since an approximation of the scale elasticity is given in proposition 3, an approximation is needed for 
\[ \sum_n \varepsilon_n'(\ln x'^{i+1}_n - \ln x'_n). \]
The quotient ratio limit of this expression is:
\[ \lim_{\ln x'^{i+1} \to \ln x'} \frac{\ln D_o(x'^{i+1}, y', t) - \ln D_o(x', y', t) - \sum_n \varepsilon_n'(\ln x'^{i+1}_n - \ln x'_n)}{\| \ln x'^{i+1} - \ln x' \|} = 0 \]
therefore a linear approximation is obtained as:
Dividing by the scale elasticity one get the base period input quantity change:

$$
\Delta \ln X' = - \frac{1}{\varepsilon} \sum_n \varepsilon_n \left( \ln x_n' - \ln x_n \right) \approx \frac{\ln D_o(x', y', t) - \ln D_o(x', y', t)}{\ln D_o(\lambda x', y')} - \ln \lambda' = \ln \lambda
$$

With a similar reasoning, for the comparison period an approximation is given by

$$
\sum_n \varepsilon_n \left( \ln x_n + \ln x_n' \right) \approx \ln D_o(x', y', t + 1) - \ln D_o(x', y', t + 1)
$$

Dividing by the comparison period elasticity one obtains the comparison period aggregate input change:

$$
\Delta \ln X_{t+1} = \frac{1}{\varepsilon_{t+1}} \sum_n \varepsilon_n \left( \ln x_n - \ln x_n' \right) \approx \frac{\ln D_o(x_{t+1}, y_{t+1}, t) - \ln D_o(x_{t+1}, y_{t+1}, t + 1)}{\ln D_o(\lambda x', y')} - \ln \lambda_{t+1} = \ln \lambda_{t+1}
$$

It has been proven that the input aggregate is the best linear approximation of the translog input aggregate. Finally, using proposition 3 and 4 together one obtains that the radial scale change (11) is the best linear approximation of the average scale change measure (4). The next propositions establish sufficient conditions for the radial scale change (11) to be equal to the average scale change (4).

**Proposition 5:** if the technology is homogeneous then the scale elasticity is constant and can be identified exactly by:

$$
\varepsilon = \frac{\ln D_o(x, y) - \ln D_o(\lambda x, y)}{\ln \lambda} = \alpha
$$

where $\alpha > 0$ is the degree of homogeneity.

**Proof:** Since an homogeneous technology satisfies: $D_o(x, y) = \frac{1}{[D(x, y)]^\alpha}$ one obtains:

$$
\varepsilon = \frac{\ln D_o(x, y) - \ln D_o(\lambda x, y)}{\ln \lambda} = \alpha \frac{\ln D_o(\lambda x, y) - \ln D_o(x, y)}{\ln \lambda} = \alpha
$$
Proposition 6: if the technology is homogeneous and inversely homothetic then the Malmquist input quantity change (9) is equal to the translog aggregate input change (3).

Proof: The input contribution is exact only if:

\[ \sum_n \varepsilon_n \left( \ln x_{n}^{t+1} - \ln x_{n}^{t} \right) + \sum_n \varepsilon_n^{\text{out}} \left( \ln x_{n}^{t+1} - \ln x_{n}^{t} \right) = \left[ \ln D_{n}(x^{t+1}, y^{t}, t) - \ln D_{n}(x^{t}, y^{t}, t) \right] + \left[ \ln D_{n}(x^{t+1}, y^{t+1}, t+1) - \ln D_{n}(x^{t}, y^{t+1}, t+1) \right] \]

Applying the quadratic identity lemma to the right hand side:

\[ \ln D_{n}(x^{t+1}, y^{t}, t) - \ln D_{n}(x^{t}, y^{t}, t) = \frac{1}{2} \sum_n \varepsilon_n \left( x^{t+1} + x^{t} \right) + \varepsilon_n \left( x^{t}, y^{t}, t \right) \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] \]

\[ \ln D_{n}(x^{t+1}, y^{t+1}, t+1) - \ln D_{n}(x^{t}, y^{t+1}, t+1) = \frac{1}{2} \sum_n \varepsilon_n \left( x^{t+1} + x^{t}, t+1 \right) + \varepsilon_n \left( x^{t}, y^{t+1}, t+1 \right) \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] \]

By substitution of these expressions and simple algebra one obtains the following equation:

\[ \sum_n \varepsilon_n \left( x^{t+1}, y^{t+1}, t+1 \right) - \varepsilon_n \left( x^{t+1}, y^{t}, t \right) \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] = - \sum_n \varepsilon_n \left( x^{t}, y^{t}, t \right) - \varepsilon_n \left( x^{t}, y^{t+1}, t+1 \right) \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] \]

If technical change is Hicks neutral and the technology output homothetic then the last equation simplifies to:

\[ \sum_n \left[ \varepsilon_n \left( x^{t+1} \right) - \varepsilon_n \left( x^{t} \right) \right] \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] = - \sum_n \left[ \varepsilon_n \left( x^{t} \right) - \varepsilon_n \left( x^{t} \right) \right] \left[ \ln x_{n}^{t+1} - \ln x_{n}^{t} \right] \]

that is trivially satisfied. This is not yet sufficient for the Malmquist input quantity change to be exact for the translog input aggregate. In fact this is true only if the scale elasticity is constant. Since homogeneity is sufficient for that, then the result is established.

The CCD Malmquist index of technical change is an exact measure of technical change when Hicks neutrality or time invariant second order coefficients hold. Otherwise it is the best linear approximation to the technical change component. For these reasons the CCD Malmquist index should be used to identify the technical change (TC) and technical efficiency change (TEC) components. The
average scale contribution is given by the radial scale change and the overall productivity index is equal to the radial productivity index of equation (7). The radial productivity index (7) is the best linear approximation of the translog productivity index proposed by Orea (2002) (when the technology is truly translog). It is defined under the minimal set of axioms defined in section 2 and it does not contain unexplained residuals (it decomposes exactly in the three sub-components of productivity change: technical change, technical efficiency change and average scale economies). In the next section the radial productivity index (7) will be used as the benchmark to evaluate the performance of other proposals. Since the radial productivity index (7) is the product of the CCD Malmquist index by a radial scale change, when the technology is locally CRS the radial scale change component is equal to 1 and the radial productivity index collapse to the CCD Malmquist index. Therefore the radial productivity index can be thought as a minimal extension of the CCD Malmquist index that includes a radial scale change (average scale economies).

5. Primal measures of productivity change: a review

In this section a number of alternative ways of defining primal productivity measures are compared. All these proposals correspond to variants of the Hicks-Moorsteen productivity index and all of them incorporate the three components discussed in the previous section (technical change, technical efficiency change and radial scale change) plus a residual that cannot be imputed to any of the previous components.

The Bjurek productivity index

Bjurek (1996) was the first proposing to measure productivity as the ratio of a Malmquist output quantity change to a Malmquist input quantity change. Bjurek (1996) proposes to use the following Malmquist indexes of output and input quantity change:

\[
Y = \left[ Y(y', y^{it+1} | x', t)Y(y', y^{it+1} | x^{it+1}, t+1) \right]^{\frac{1}{2}}
\]

\[
X = \left[ X(x', x^{it+1} | y', t)X(x', x^{it+1} | y^{it+1}, t+1) \right]^{\frac{1}{2}}
\]

These Malmquist quantity changes use observed input-output vectors as conditioning vectors. The ratio of the output quantity change and the input quantity change just introduced gives the Bjurek measure of productivity change: \( B(x', y', x^{it+1}, y^{it+1}) = \frac{Y}{X} \). This productivity index is the geometric mean of a base

\[\text{As emphasized by Briec and Kerstens (2010) the Bjurek index is the only productivity index that satisfies the determinateness axiom.}\]
period index and a comparison period index. Dividing the base period Bjurek index by the base period radial productivity index (7) one obtains the residual of the Bjurek index:

$$\frac{B'(t)}{RPI'(t)} = \frac{D_o(\bar{x}, \bar{y}^{t+1}) D_i(x', y', t) D_o(x', y', t) D_i(x', y', t) D_i(x', y', t)}{D_o(x', y', t) D_i(x', y', t) D_o(x', y', t) D_i(x', y', t) D_i(x', y', t)}$$

This is the product of two components. The first, defined by the output distance functions, is equal to one if and only if the technology is locally output homothetic; therefore it is a measure of deviation from the output homothetic assumption. The second component, defined by input distance functions, is equal to one when the technology is CRS or input homothetic. These residuals are ways of testing for homotheticity as shown by Primont and Primont (1994, 1996) and do not have a correspondence with any of the productivity components identified in the previous sections. For the comparison period index, dividing by the comparison period radial productivity index:

$$\frac{B'^{t+1}(t)}{RPI'^{t+1}(t)} = \frac{D_o(x^{t+1}, y^{t+1}, t+1) D_i(x', y', t+1) D_o(x', y', t+1) D_i(x', y', t+1) D_i(x', y', t+1)}{D_o(x', y', t+1) D_i(x', y', t+1) D_o(x', y', t+1) D_i(x', y', t+1) D_i(x', y', t+1)}$$

The components of this residual have the same interpretation of the base period residual. The component based on output distance functions is equal to one if and only if the comparison period technology is output homothetic. The component based on input distance functions is equal to one if the comparison period technology is CRS or input homothetic.

The Generalized Malmquist productivity index

Grifell-Tatje and Lovell (1999) (GTL from now on) proposed a generalized Malmquist productivity index which is defined as the product of the output oriented CCD Malmquist index by a scale change factor. The authors follow a bottom-up procedure for the building of the index, although the base period index ends up to be the following variant of the Hicks-Moorsteen index:

$$GTL = \frac{Q(x', y^{t+1} | x^{t+1}, t)}{Q(x', x^{t+1} | y', t)}$$

Here the subscript “c” means that the input distance functions are computed against the virtual CRS cone technology. This index is the ratio of a Malmquist output quantity change to a Malmquist input quantity change (computed against a virtual CRS cone technology). The index deviates from the Bjurek (1996) proposal in the fact that the input distance functions are here computed against the virtual CRS cone.
technology\textsuperscript{16} and the conditioning input vector for the Malmquist output quantity change is the comparison period one. Dividing the base period index by the base period radial productivity index one obtains the residual for the GTL index:

\[
\frac{GTL}{RPI} = \frac{D_b(x', y', t)}{D_b(x'^{\prime\prime}, y', t)} \frac{D_b(x'^{\prime\prime}, y', t)}{D_b(x', y', t)}
\]

An intuitive way of thinking this residual is imagining a movement along a pre-assigned isoquant. When \(y' = y'^{\prime\prime} = \bar{y}'\) and \(D_b(x', y', t) = D_b(x'^{\prime\prime}, y', t) = 1\) the input vectors belong to the same isoquant. It is easy to check that in this eventuality the residual in the GTL index deviates, in general, from one. This means that the GTL productivity index is not invariant to movement along indifference surfaces. It should be noted that the structure of the RPI follow the idea presented in Grifell-Tatje and Lovell (1999) of getting the primal measure of productivity as the product of the CCD Malmquist by a scale change component. The difference is that the GTL scale change component is not a measure of average scale economies. GTL define the scale change component as a ratio of output scale efficiency measures:

\[
\frac{SE_o(x_{it+1}, y_{it}, t)}{SE_o(x_{it}, y_{it}, t)}
\]

This measure of the scale contribution incorporates a residual effect; decomposing it in the following way

\[
\frac{SE_o(\tilde{x} x_{it}, y_{it}, t)}{SE_o(x_{it}, y_{it}, t)} \frac{SE_o(x_{it+1}, y_{it}, t)}{SE_o(\tilde{x} x_{it}, y_{it}, t)}
\]

it is easy to see that the first component is a radial scale change (average scale change) and the second one is an unexplained residual. In fact since \(\tilde{x}'\) has been chosen such that \(D_b(\tilde{x} x', y', t) = D_b(x'^{\prime\prime}, y', t)\), it follows that the index is not invariant to movement along an indifference surface. The residual is a signal of this lack of indifference invariance. The comparison period index is:

\[
GTL^{\prime\prime} = \frac{Q(y_{it}, y_{it+1} | x_{it}, t + 1)}{X_r(x_{it}, x_{it+1} | y_{it+1}, t + 1)}
\]

which is the same as the Bjurek comparison period index but for the fact that the input distance function is here computed against a virtual CRS cone technology. The ratio of this index to the comparison period radial productivity index gives:

\textsuperscript{16} It is interesting to note that from a theoretical perspective, productivity indexes defined using the virtual CRS technology are meaningful only if the associated virtual technology exists. This means that they are defined under less general circumstances than the radial productivity measure.
\[
\frac{GTL^{t+1}}{RPI_t} = \frac{D_{oc}(x^t, y^{t+1}, t + 1)}{D_{oc}(x^{t+1}, y^{t+1}, t + 1)} \frac{D_{oc}(x^{t+1}, y^{t+1}, t + 1)}{D_{oc}(x^t, y^{t+1}, t + 1)}
\]

The comments about this residual are similar to the ones of the base period residual and correspond to lack of invariance for movements along indifference surface (a pre-assigned input isoquant).

**The FGNZ index**

In their influential paper Fare et al (1994) (FGNZ from now on) proposed to measure productivity change by evaluating the CCD Malmquist index using a CRS virtual enlargement of the production set\(^{17}\).

This measure of productivity has been proposed or discussed in a theoretical framework (without the pretension of being exhaustive) by: Fare et al (1994), Ray and Desli (1997), Balk (2001), Grosskopf (2003), Lovell (2003), Balk (2004). The base period index is\(^{18}\):

\[
FGNZ_t = \frac{D_{oc}(x^{t+1}, y^{t+1}, t)}{D_{oc}(x', y', t)}
\]

Since under CRS the output distance function is equal to the inverse of the input distance function the last index can be written as:

\[
FGNZ_t = \frac{D_{oc}(x^{t+1}, y^{t+1}, t) \cdot D_{oc}(x^{t+1}, y^{t+1}, t + 1)}{D_{oc}(x', y', t) \cdot D_{oc}(x', y', t + 1)} = \frac{Q_t(y', y^{t+1} | x^{t+1}, t)}{X_t(x', x^{t+1} | y', t)}
\]

The denominator of this expression is equal to the one proposed by Grifell-Tatje and Lovell (1999). The difference is that the numerator is now computed against a virtual CRS cone technology. This index can still be regarded as a variant of the Hicks-Moorsteen approach, being a ratio of a Malmquist output quantity change to a Malmquist input quantity change. The comparison period index is:

\[
FGNZ^{t+1} = \frac{D_{oc}(x^{t+1}, y^{t+1}, t + 1)}{D_{oc}(x', y', t + 1)}
\]

The index can be written as:

\(^{17}\)This is the most widely used measure in empirical works. In fact the FGNZ index and its decomposition are incorporated in the DEAP free software developed by Tim Coelli.

\(^{18}\)Here as well the productivity measure relies on the existence of a meaningful virtual CRS cone technology. Therefore the minimal axioms of production theory are not sufficient for a measure like that to exist.
This is the Grifell-Tatje and Lovell (1999) comparison period denominator with a numerator computed against the virtual CRS technology. Dividing the base period index by the base period radial productivity index one obtains:

\[
FGNZ^{t+1} = \frac{D_n(x^{t+1}, y^{t+1}, t+1)}{D_n(x', y', t+1)} \frac{D_n(x', y^{t+1}, t+1)}{D_n(x', y'^{t+1}, t+1)} = \frac{Q(x', y'| x', t+1)}{X(x', x^{t+1}, y^{t+1}, t+1)}
\]

The two components have the same structure. Since output scale efficiency is homogenous of degree zero in outputs and input scale efficiency is homogeneous of degree zero in inputs, these two residuals measure the displacement of the virtual CRS technology due to a change in input and output mix. This is a typical situation where the productivity measure deviates from one with movement along a pre-assigned isoquant. In fact, if one assumes no technical change, full technical efficiency, \( y' = y'^{t+1} \) and \( D_i(x', y', t) = D_i(x'^{t+1}, y', t) \) (this is a movement along the isoquant determined by the fixed output vector), the FGNZ index will deviate from one. This means that the index is not invariant to movements along an indifference surface. Both these measures are movements of a hypothetical technology and they do not have relation to productivity gains in the real technology. Dividing the comparison period index by the comparison period radial productivity measure one obtains:

\[
\frac{FGNZ^{t+1}}{RPI^{t+1}} = \frac{SE_n(x^{t+1}, y^{t+1}, t)}{SE_n(x'^{t+1}, y'^{t+1}, t)} \frac{SE_n(x', y'^{t+1}, t)}{SE_n(x'^{t+1}, y'^{t+1}, t+1)}
\]

The comparison period residuals have the same interpretation of the base period residuals and they show lack of invariance to movement along indifference surfaces.

### 6. Empirical Illustration

An interesting question which arises when using these indexes in an empirical context is the magnitude of the residuals embedded in the Bjurek, FGNZ and GTL productivity indexes. The procedures outlined in the previous sections have been applied to a real dataset to investigate the empirical relevance of the residuals. In this empirical illustration it is shown that the residuals contained in the three main productivity measures (FGNZ, GTL and Bjurek) can be severe and dominate the productivity formula. This means that these residuals do not have the nature of second order residuals and they do not vanish.
under specified assumption on the data generating process. Computation of the distance functions has been done using the Data Envelopment Analysis (DEA) approach. All the linear programming problems associated to the previous index numbers are developed in the Appendix. All the computations use Matlab code which is available on request. Data come from Coelli and Rao (2005) and they are a collection of 5 inputs (labour, capital, land, fertilizers and livestock) and 2 outputs (crop and animals) for 88 countries in the time span 1970-2000 (agricultural sector). The radial productivity index (7) is computed along with: the CCD Malmquist index; the FGNZ index; the Bjurek index; the GTL index; the average scale economies. In Table 1 geometric mean for each country and for each index is shown (for the full sample of years).

Productivity growth measured by the radial productivity index (RPI) is, on average, almost negligible (1.002). This tiny growth is the result of a positive growth in the CCD Malmquist index (1.010) and a negative contribution of average scale economies (0.992). Differences among countries are quite large with Nigeria showing the worst performance (0.932) and Denmark the best one (1.033). In 78% of cases the indexes return a number on the same side of unity although the magnitude of productivity change has large variations. In the case of observations with negligible average scale economies components (in the range 0.999-1.001), it is possible to check that the radial index equalizes the CCD Malmquist index (that is an appropriate measure of productivity change in this special circumstance). On the contrary, for the same observations, the other indexes of productivity show remarkable differences due to residual effects. In the case, for example, of Angola the average scale economies contribution to productivity change was almost null (1.000). As expected the CCD Malmquist and the radial productivity measure coincides (1.015 and 1.016) while the other two indexes greatly underestimate productivity growth of almost 1/3 (1.005 for both indexes).

An interesting thing to do is comparing cases where the four index numbers return signals on a different side of the unity. Figure 1 reports a bar plot for the countries where the indexes does not lie on the same side of unity (the figure shows rate of growth instead of growth factors). As the case of Nigeria testifies, also in cases with large productivity variations the signal given by the different index numbers can be controversial. Nigeria de-growth in productivity at a rate of 6.8% (RPI) was approximately identified well by the Bjurek index (4.4%). On the contrary, both the GTL and FGNZ indexes returns positive rate of productivity growth for Nigeria, therefore signalling a strong effect of their residual components. Tables from 2 to 6 report the percentage of opposite signals (different side of unity) for all the couples. These tables have been built using year by year and country by country information (the values are not computed using the average values of table 1, but the full sample). While table 1 report the
percentage of cases with opposite signal, the other tables use a bandwidth around the unity: the percentage is calculated based on the number of times that the two index numbers compared fall outside this interval and on opposite sides. The bandwidth has been selected to be 0.005, 0.01, 0.015 and 0.02. As these tables show also when the productivity signal is strong the percentage of cases with index numbers lying on different side of the unity is appreciable. This means that the residual effects can be so strong to overcome the productivity signal. Also in cases where the productivity signal was strong (bandwidth = 0.02) the FGNZ and RPI give signals on the opposite side of the interval in 5.5% of cases. This means that the residuals do not vanish either when the productivity growth signal is very strong.

7. Conclusion

A primal index of productivity change has been introduced which decomposes exactly in three components: technical change, technical efficiency change and average scale economies. The productivity index is defined under very weak assumptions on the production set and it does not require special assumptions about the functional form of the output distance function. Since the technical change and technical efficiency change components sum up to the CCD Malmquist productivity index, the new index is a minimal extension that incorporates average scale economies. It has been emphasized that average scale economies cannot, in general, be measured accurately as the change in the scale efficiency of the production process. Previous attempts of doing so produced productivity indexes which decompose in the aforementioned three components plus residuals arising from lack of invariance to movement along indifference surfaces (isoquants). It has been shown with an empirical illustration on real data that the magnitude of the residuals components embedded in previous productivity measures can be very large. For observations with negligible average scale economies components the radial productivity index returns the same number of the CCD Malmquist index, while the other proposals (in some cases) deviates remarkably from these figures. This study suggests that the radial productivity measure is an appropriate measure of productivity growth under very general conditions and its interpretation and decomposition are very easy to run and understand. Since the radial productivity index here introduced is the best linear approximation of the translog productivity index (growth accounting), it represents the best nonparametric companion of the translog growth accounting methodology.

Acknowledgements

I would like to thank C.J. O’Donnell, T. J. Coelli, D.S. Prasada Rao, E. Grifell-Tatje, E. Diewert and participants to a seminar at University of Queensland for useful comments on a preliminary version of
Comments and encouragements by Knox Lovell were an invaluable tool in advancing my knowledge of the topic. The usual disclaimers apply.
APPENDIX 1 – Tables and Figures

Table 1 – Annual geometric average for all countries and all indexes

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Figure 1 – Annual average growth rate for selected countries (all indexes)
### Table 2 - Interval around the unity = 0.000

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<th>FGNZ</th>
<th>GTL</th>
<th>RPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bjurek</td>
<td>0.0</td>
<td>10.9</td>
<td>11.2</td>
<td>4.3</td>
</tr>
<tr>
<td>FGNZ</td>
<td>0.0</td>
<td>7.5</td>
<td></td>
<td>9.7</td>
</tr>
<tr>
<td>GTL</td>
<td>0.0</td>
<td>8.1</td>
<td></td>
<td>8.1</td>
</tr>
<tr>
<td>RPI</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>

### Table 4 - Interval around the unity = 0.010

<table>
<thead>
<tr>
<th>Percentage of Opposite Signals</th>
<th>Bjurek</th>
<th>FGNZ</th>
<th>GTL</th>
<th>RPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bjurek</td>
<td>0.0</td>
<td>9.4</td>
<td>10.0</td>
<td>3.1</td>
</tr>
<tr>
<td>FGNZ</td>
<td>0.0</td>
<td>6.7</td>
<td></td>
<td>8.3</td>
</tr>
<tr>
<td>GTL</td>
<td>0.0</td>
<td>7.1</td>
<td></td>
<td>7.1</td>
</tr>
<tr>
<td>RPI</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>

### Table 5 - Interval around the unity = 0.015

<table>
<thead>
<tr>
<th>Percentage of Opposite Signals</th>
<th>Bjurek</th>
<th>FGNZ</th>
<th>GTL</th>
<th>RPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bjurek</td>
<td>0.0</td>
<td>8.1</td>
<td>8.6</td>
<td>2.1</td>
</tr>
<tr>
<td>FGNZ</td>
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<td>5.5</td>
<td></td>
<td>6.8</td>
</tr>
<tr>
<td>GTL</td>
<td>0.0</td>
<td>5.3</td>
<td></td>
<td>5.3</td>
</tr>
<tr>
<td>RPI</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>

### Table 6 - Interval around the unity = 0.020

<table>
<thead>
<tr>
<th>Percentage of Opposite Signals</th>
<th>Bjurek</th>
<th>FGNZ</th>
<th>GTL</th>
<th>RPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bjurek</td>
<td>0.0</td>
<td>6.6</td>
<td>6.7</td>
<td>1.6</td>
</tr>
<tr>
<td>FGNZ</td>
<td>0.0</td>
<td>4.5</td>
<td></td>
<td>5.5</td>
</tr>
<tr>
<td>GTL</td>
<td>0.0</td>
<td>4.3</td>
<td></td>
<td>4.3</td>
</tr>
<tr>
<td>RPI</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>
APPENDIX 2 – DEA Linear Programs

Data are collected into two matrices: the output matrix \( Y^t = [y_{mk}^t] \) and the input matrix \( X^t = [x_{mk}^t] \). To compute the CCD Malmquist index the following linear programs (in their dual representation, see Thanassoulis et al, 2008) have to be computed (DEA):

\[
\frac{1}{D_o(x^{rel}, y^{rel}, t+1)} = \min (v x^{rel} + w)
\]

\[
\text{st} \quad v X^{rel} + e w - u Y^{rel} \geq 0
\]

\[
u y^{rel} = 1
\]

\( u, v \geq 0 \), \( \text{wis free} \)

\[
\frac{1}{D_o(x', y', t+1)} = \min (v x' + w)
\]

\[
\text{st} \quad v X' + e w - u Y' \geq 0
\]

\[
u y' = 1
\]

\( u, v \geq 0 \), \( \text{wis free} \)

\[
\frac{1}{D_o(x^{rel}, y^{rel}, t)} = \min (v x^{rel} + w)
\]

\[
\text{st} \quad v X^{rel} + e w - u Y^{rel} \geq 0
\]

\[
u y^{rel} = 1
\]

\( u, v \geq 0 \), \( \text{wis free} \)

\[
\frac{1}{D_o(x', y', t)} = \min (v x' + w)
\]

\[
\text{st} \quad v X' + e w - u Y' \geq 0
\]

\[
u y' = 1
\]

\( u, v \geq 0 \), \( \text{wis free} \)

To compute the FGNZ index the previous linear programs have to be solved imposing \( w = 0 \) (this is equivalent to a cone CRS envelopment of the data). The Bjurek index requires these two additional linear programs for computing the Malmquist output quantity change:
Moreover one needs to compute the Malmquist input quantity indexes solving the following linear programs:

\[
\frac{1}{D_s(x^{t+1},y^t, t+1)} = \min(vx^{t+1} + w)
\]

\[
\text{st } \quad vX^{t+1} + ew - uY^{t+1} \geq 0
\]

\[
uy^t = 1
\]

\[
u, v \geq 0, \quad \text{wia free}
\]

\[
\frac{1}{D_s(x',y^{t+1}, t)} = \min(vx' + w)
\]

\[
\text{st } \quad vX' + ew - uY' \geq 0
\]

\[
uy^{t+1} = 1
\]

\[
u, v \geq 0, \quad \text{wia free}
\]
Finally, being the radial productivity index equal to the CCD index by the radial scale change, it is sufficient to solve the following two additional linear programs:

\[
\frac{1}{D_r(x^{rt}, y', t)} = \min (vx^{rt} + w) \\
\text{st } \begin{align*}
v'X' + ew - uY' &\geq 0 \\
u' &\leq 1 \\
u, v &\geq 0, \text{ is free}
\end{align*}
\]

\[
\frac{1}{D_r(x', y'^{rt}, t + 1)} = \min (vX' + w) \\
\text{st } \begin{align*}
vX'^{rt} + ew - uY'^{rt} &\geq 0 \\
u &\leq 1 \\
u, v &\geq 0, \text{ is free}
\end{align*}
\]
References


Balk B.M. (2004), The many decompositions of productivity change, Erasmus Research Institute of Management, Erasmus University Rotterdam

Briec W., Kerstens K. (2010), The Hicks-Moorsteen productivity index satisfies the determinateness axiom. The Manchester School. doi: 10.1111/j.1467-9957.2010.02169.x


